

# Painlevé Classification Of Polynomial Ordinary Differential Equations Of Arbitrary Order And Second Degree

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## Abstract

The problem of Painlevé classification of ordinary differential equations lasting since the end of XIX century saw significant advances for the limited equation order, however not that much for the equations of higher orders. In this work we propose the complete Painlevé classification for ordinary differential equations of the arbitrary order with right-hand side being a quadratic form on the dependent variable and all of its derivatives. The total of seven classes of the equations with Painlevé property have been found. Five of them having the order up to four are already known. Sixth one of the other up to five also appears to be integrable in the known functions. While the only seventh class of the unrestricted order appears to be linearizable. The classification employs a novel general necessary condition for the Painlevé property proven in the paper, potentially having a broader application for the Painlevé classification of other types of ordinary differential equations.

## Introduction

The Painlevé classification is one of the long-lasting problems of analytic theory of differential equations rooted in the end of XIX century. In spite of numerous achievements over the last more than hundred years on the classification of the equations of particular limited order (mainly up to fourth), the general problem for the higher order equations remains unsolved. This paper contributes towards solution of the arbitrary-order problem by building the complete Painlevé classification for the broad class of equations with the only constraint on the equation's degree but free of any order limitations.

The Painlevé classification of the non-linear polynomial ordinary differential equations

$$w^{(n)} = P(w^{(n-1)}, w^{(n-2)}, \dots, w, z), \quad (1)$$

where  $P$  is a polynomial in  $w$  and its derivatives with coefficients locally analytic in  $z$ , is known for the order  $n \leq 4$ . For the order  $n = 1$  the well-known necessary and sufficient condition of the Painlevé property for the equation (1) is  $\deg P = 2$  (see for instance [1], chapter XIII). For the order  $n = 2$  classification has been built in the classical works of Painlevé and Gambier (see for instance [1], chapter XIV). For the order  $n = 3$  classification has been started in the famous work of Chazy [2] and recently completed by C.Cosgrove [3]. Finally for the order  $n = 4$  the problem was solved by C.Cosgrove [4,5]. And although complete Painlevé classification has been successfully completed for certain algebraic classes of equations of the arbitrary order, such as binomial-type equations [6–9] and arbitrary algebraic equations that do not depend on the derivatives of order  $n - 1$  and  $n - 2$  [9–12], the classification of analytically more simple polynomial-class ordinary differential equations of order  $n \geq 5$  is not yet accomplished in the general case.

Here we consider a class of polynomial ordinary differential equations of arbitrary order  $n \geq 2$ , but restricted by the degree of the right-hand side. Let  $P$  be a quadratic form in  $w$  and its derivatives, i.e.

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consider the equation

$$w^{(n)} = \sum_{n-1 \geq k \geq j \geq 0} a_{k,j}(z) w^{(k)} w^{(j)} + \sum_{j=0}^{n-1} b_j(z) w^{(j)} + c(z), \quad (2)$$

where  $a_{k,j}, b_j, c$  are functions in  $z$  analytic in the certain complex domain  $U$ . Without loss of generality assume  $a_{k,j}(z) \equiv a_{j,k}(z)$ . We exclude the order  $n = 1$  since all Ricatti equations, as mentioned afore, certainly possess the Painlevé property.

First give the rigorous definition of the main concept - the Painlevé property - for the considered equations (1) inline with it's classical understanding [1, 13] but distinguishing commonly mixed concepts of freedom from movable branch points and non-polar singularities.

**Definition 1.** Consider an arbitrary solution  $w = w(z)$  of the equation (2) analytic in the neighborhood of some point  $z^* \in U$  and a path  $\Gamma$  with the beginning in  $z^*$  along which all the coefficients of (2) can be analytically continued while the analytical continuation of  $w(z)$  comes to a singularity. Such a singularity of  $w(z)$  is called movable singularity of the considered solution  $w = w(z)$ .

**Definition 2.** The equation (2) is called to possess the Painlevé property if the equation's solutions are single-valued near all of their movable singularities. The equation (2) is called to possess the strong Painlevé property if all (if any) movable singularities of it's solutions are poles.

The equations possessing the strong Painlevé property according to the definition 2 have been also introduced in [14] as the equations of Painlvé-type.

Our goal in the present paper is to find all of the equations of class (2) possessing the strong Painlevé property.

## 1 First necessary condition

Introduce a constant  $B = n - \max\{k + j : a_{k,j}(z) \not\equiv 0\}$ . This characteristic indicates the possible order of a movable pole for equation's (2) solutions. According to the theorem 4 [15] if the initial equation (2) possesses the Painlevé property and moreover the strong Painlevé property, the number  $B$  could be only 1 or 2. The corresponding two possible forms of the equation (2) with the Painlevé property are:

$$w^{(n)} = \sum_{k=[n/2]}^{n-1} a_{k,n-1-k}(z) w^{(k)} w^{(n-1-k)} + \sum_{k+j < n-1} a_{k,j}(z) w^{(k)} w^{(j)} + \sum_{j=0}^{n-1} b_j(z) w^{(j)} + c(z), \quad (3)$$

$$w^{(n)} = \sum_{k=[(n-1)/2]}^{n-2} a_{k,n-2-k}(z) w^{(k)} w^{(n-2-k)} + \sum_{k+j < n-2} a_{k,j}(z) w^{(k)} w^{(j)} + \sum_{j=0}^{n-1} b_j(z) w^{(j)} + c(z), \quad (4)$$

where at least one of the coefficients  $a_{k,j}$ , for which  $n - (k + j) = B$ , is not identically equal to zero. Moreover with respect to the theorem 3 [15] at least one of such coefficients having  $k \geq n - 2$  or  $l \geq n - 2$  is not identically equal to zero as the equation's learning terms should include  $w^{(n-1)}$  or  $w^{(n-2)}$ .

## 2 Improved resonance condition

According to [15, 16] the equation (1) always admits solutions with movable singularities. And if the equation possesses the strong Painlevé property these singularities are poles. Below we construct one general necessary condition that the equation should satisfy in order for that to hold.

First of all if the equation (1) admits a solution with movable singularity in a certain point  $z = z_0$  being a pole, their should exist a Laurent expansion of the form

$$w = \sum_{j=0}^{\infty} q_j (z - z_0)^{j-p}, \quad (5)$$

converging in a certain deleted neighborhood of  $z = z_0$ , where  $p$  is an integer order of the pole, while  $a_j$  are complex coefficients, provided that  $q_0 \neq 0$ . Further we first reproduce the well-known technique of the Painlevé test [17] in a rigorous way and also prove one additional necessary condition applied to the equation's so called resonance numbers.

Call some of the terms of the equation (1) leading with respect to  $(p, z_0)$  if their coefficients are nonzero in  $z_0$  and these terms produce the maximum order of the pole in  $z = z_0$  after substitution of the expression (5) in the equation (1) compared to all other terms of the equation. Denote the set of those leading terms by  $L(p, z_0)$ .

Denote by  $S$  the set of points in the complex domain being a zero for at least one of equation's (1) (which could be written in the form (2)) coefficients. Note that the set of leading terms  $L(p, z_0)$  does not depend on  $z_0$  for  $z_0 \notin S$  and denote it by  $\hat{L}(p)$ . For  $z_0 \in S$  the set  $L(p, z_0)$  is a subset of  $\hat{L}(z_0)$  if at least one of the coefficients of the terms from  $\hat{L}(p)$  is nonzero. In the opposite case  $L(p, z_0)$  is the set of new terms producing lower pole orders in  $z_0$ .

Substituting series (5) with undefined coefficients  $q_j$  into the equation (1) or (2) one should obtain an algebraic equation

$$T(q_0) = 0 \quad (6)$$

for determination of  $q_0$  (so called determining equation) and the following recurrent equations for determining each further  $q_j$  through the earlier defined  $q_0, q_1, \dots, q_{j-1}$ :

$$R(j, q_0)q_j = Q_j(q_0, q_1, \dots, q_{j-1}), \quad (7)$$

where  $T$ ,  $R$  and  $Q_j$  are polynomials in all variables.

The polynomials  $T$  and  $R$  depend only on the leading terms  $L(p, z_0)$  (or  $\hat{L}(p)$  in case  $z_0 \notin S$ ), while the polynomials  $Q_j$  together depend on all the terms of the equation (1). For  $z_0 \notin S$  the coefficients of the polynomials  $T$  and  $R$  are polynomial expressions in all the values of the coefficients of the leading terms from  $\hat{L}(p)$  in the point  $z_0$ , while the coefficients of  $Q_j$  are polynomial expressions in  $z_0$  and the values of coefficients of (2) and their derivatives in the point  $z_0$ . That is why for  $z_0 \notin S$  the determining equation (6) and the equations (7) have the same general form for all  $z_0$  while coefficients of polynomials  $T, R, Q_j$  are analytic in  $z_0$ .

Since  $q_0 \neq 0$  in (5) the determining equation should admit nonzero roots and so there should exist at least two leading terms  $L(p, z_0)$  for each  $z_0$  being the position of a movable pole of order  $p$ . Generally the determining equation (6) may have several solutions  $q_0 = q_0(z_0)$  being analytic functions in  $z_0$  for all  $z_0 \notin S$  probably except of a countable set of algebraic singularities.

Starting from the choice of one of the roots  $q_0$  of the determining equation, one could use the equations (7) to find all of the other  $q_j$  except of those who's indexes  $j = r$  are the roots of the equation

$$R(r, q_0) = 0, \quad (8)$$

called resonance equation for the given choice of  $q_0$ . Its roots are called resonance numbers (or simply resonances) of the equation (1) for the selected  $q_0$ . One of those resonances is always  $r = -1$  [17] and is called trivial. The rigorous proof of this fact could be found in [13] (page 126-127).

The degree of the polynomial  $R$  in  $r$  does not exceed the maximum order of the derivative of  $w$  contained among the leading terms  $L(p, z_0)$ . So the number of resonances other than trivial  $r = -1$  is no more than  $n - 1$ , while the maximum number of  $n - 1$  is achieved only if the major derivative  $w^{(n)}$  is contained among the leading terms  $L(p, z_0)$ .

Denote the set of resonances corresponding to the given set of  $p, z_0, q_0$  by  $r(p, z_0, q_0)$ . As we have shown this set is completely defined by the choice of  $p, z_0, q_0$ . Call the set of numbers  $p, z_0, q_0$  the initial characteristics of the movable pole of the solution  $w = w(z)$ . So one can say that the certain movable pole of the certain solution possesses the set of resonances  $r(p, z_0, q_0)$  defined by the initial characteristics  $p, z_0, q_0$  of this movable pole.

In that terms the well-known resonance condition for the Painlevé property [13] can be formulated as following: for each  $z_0$  and for every possible pair of  $p, q_0$  (i.e. every pair  $p, q_0$  such that  $L(p, z_0)$  contains

more than one term,  $q_0$  satisfies the corresponding determining equation and solutions of the equation (1) with leading asymptotic behavior  $w \sim q_0(z - z_0)^{-p}$  really exist) all of the roots of resonance equations (8) with positive real part should be real integer. Further in [13] (page 132) for instance it is stated that in fact all the roots of (8) should be distinct integers.

The positive integer resonances are values of indexes  $j$  for which the value of the coefficient  $q_j$  can be arbitrary. The further necessary conditions for the Painlevé property could be obtained by analyzing the equations (7) for indexes  $j$  being a positive integer resonance. Such equations take the form  $Q_j(q_0, q_1, \dots, q_{j-1}) = 0$  and give additional conditions which should be satisfied with respect of the earlier obtained expressions for the coefficients  $q_0, q_1, \dots, q_j$ . These conditions are called resonance conditions.

There exist a number of works devoted to the Painlevé analysis of the certain equations possessing negative resonances [13] (page 139-140), [18–21], however it is still not quite clear what information could negative resonances give for the Painlevé test in general.

It is also mentioned in [17] that if for every pair of  $p, q_0$  resonance roots do not contain  $n-1$  nonnegative distinct integers then (5) does not represent a general solution. It hints that perhaps general solution has a more complex shape and Painlevé property does not hold in such a case, however rigorous proof of this fact is not provided as for the author's knowledge. Below we'll fill this formal gap.

For a movable pole of order  $p$  in point  $z_0$  with leading coefficient  $q_0$  call the set of resonances  $r(p, z_0, q_0)$  complete if, besides the trivial resonance  $-1$ , it consists of  $n-1$  distinct non-negative integers.

**Theorem 1.** *If the equation (1) possesses the strong Painlevé property and admits a solution with movable pole in a certain point  $z = z_1$  then for any deleted neighborhood  $V$  of  $z_1$  there exists a solution of the equation (1) with a movable pole  $z_0 \in V$  possessing complete set of resonances.*

In other words the theorem 1 states that negative resonances other than trivial could exist but not for all movable singularities of solutions of the equation (1) with strong Painlevé property. Moreover positions of movable poles of solutions can not be isolated and near each location of a movable pole one could always find other movable poles of other solutions possessing complete set of resonances.

**Proof of the theorem 1.** Suppose the opposite case i.e. let  $w = w_0(z)$  be a solution of the equation (1) with a movable pole in  $z = z_1$  and suppose that for any deleted neighborhood  $V$  of  $z_1$  all movable poles of solutions of the equation (1) located in  $V$  possess no more than  $n-2$  distinct non-negative integer resonances.

Consider an arbitrary small closed contour  $\Gamma$  such as all the coefficients of the equation 1 are analytic in  $G = \text{int}\Gamma$  (closed interior of  $\Gamma$ ) while  $z = z_0$  is the only singularity of the solution  $w = w_0(z)$  in  $G$ . Select an arbitrary point  $z = z^*$  on  $\Gamma$  and consider a local representation for the general solution of equation (1) as  $w = \phi(z, \lambda)$ , where  $\lambda = (w^0, w^1, \dots, w^{n-1})$  is the set of parameters for the corresponding Cauchy initial value problem  $w^{(j)}(z^*) = w^j$ , while function  $\phi$  is analytic for  $z \in \Gamma$  and  $\lambda$  from a certain  $n$ -dimensional neighborhood  $V$  of  $\lambda_0 = (w_0(z^*), w'_0(z^*), \dots, w_0^{(n-1)}(z^*))$ . This way  $\phi(z, \lambda_0) = w_0(z)$ . For  $\lambda$  close enough  $\lambda_0$  solutions  $w = \phi(z, \lambda)$  should also possess movable singularities inside  $\Gamma$  (being poles since the equation possess strong Painlevé property) as otherwise function  $w_0(z)$  would be analytic inside  $\Gamma$  as a limit of the sequence of analytic functions). Without loss of generality assume that it holds for all  $\lambda \in V$ , otherwise simply narrow  $V$  accordingly.

Then  $\phi$  could be represented in the following form

$$\phi(z, \lambda) = h(z, \lambda) + \varphi(z, \lambda), \quad (9)$$

where  $\varphi(z, \lambda) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\phi(\zeta, \lambda)}{\zeta - z} d\zeta$ . Function  $\varphi(z, \lambda)$  by definition is analytic in  $G \times V$ , while  $h(z, \lambda) = \phi(z, \lambda) - \varphi(z, \lambda)$  for each  $\lambda \in V$  is analytic in  $z$  on  $\Gamma$  and outside it including infinity, having the same poles inside  $\Gamma$  as  $\phi(z, \lambda)$  has. This way function  $h$  is a rational function in  $z$  with coefficients analytic for  $\lambda \in V$ .

Let  $\tau(\lambda)$  be the number of poles of function  $h(z, \lambda)$  in  $z$ . Function  $\tau$  is limited on  $V$  since  $h$  is rational. Consider  $\tau^* = \max_{\lambda \in V} \tau(\lambda)$  and a certain  $\lambda^* \in V$  such as  $\tau(\lambda^*) = \tau^*$ . Then one can show that each of the poles  $z_1 \in G$  of the solution  $\phi(z, \lambda^*)$  should possess a complete set of resonances.

Indeed there exist a contour  $\Gamma_1$  surrounding  $z_1$  such as for any  $\lambda$  from a certain neighborhood  $V_1$  of

$\lambda^*$ ,  $\phi(z, \lambda)$  posses just one single pole inside  $\Gamma_1$  as otherwise  $\tau^* < \max_{\lambda \in V} \tau(\lambda)$  (since for any  $\lambda$  close enough to  $\lambda^*$  function  $h$  should also have at least one pole near each of all other locations of poles of  $h(z, \lambda^*)$ ). Now similar to (9) represent the general solution of (1) as

$$\phi(z, \lambda) = h_1(z, \lambda) + \varphi_1(z, \lambda), \quad (10)$$

where  $\varphi_1$  is analytic for  $z$  on and inside  $\Gamma_1$  and  $\lambda \in V_1$ , while  $h_1$  is the rational function in  $z$  with coefficients analytic in  $\lambda$  having a single pole in  $z$ . Obviously (10) is a local representation of the general solution for the equation (1) in the form of Laurent series around a movable pole. While if the resonance set of the solution  $\phi(z, \lambda^*)$  corresponding to the pole in  $z_1$  is not be complete, having only  $n_1 < n - 1$  distinct positive integer resonance numbers, it would mean that all the coefficients of (10) could be uniquely defined through  $m + 1 < n$  arbitrary coefficients, i.e. (10) would not represent the general solution of the equation (1). Obtained contradiction completes the proof of the theorem 1.

### 3 The resonance equation

Now apply the resonance condition of theorem 1 to the two possible cases (3) and (4) of the equation (2). First from the equation (6) find the possible major coefficient  $q_0$  for the movable pole in the point  $z = z_0$  in the form  $q_0 = 1/f(z_0)$ , where

$$f(z) = - \sum_{k=[n/2]}^{n-1} a_{k, n-1-k}(z) k! (n-1-k)! / n!$$

for the equation (3) and

$$f(z) = \sum_{k=[(n-1)/2]}^{n-2} a_{k, n-2-k}(z) (k+1)! (n-1-k)! / (n+1)!$$

for the equation (4). Then obtain the resonance equations (8) for the cases (3) and (4) in the corresponding forms:

$$0 = R(r) = \prod_{t=0}^{n-1} (r-1-t) - \sum_{k=[n/2]}^{n-1} \frac{a_{k, n-1-k}(z_0)}{f(z_0)} \left( (-1)^{n-1-k} (n-1-k)! \prod_{t=0}^{k-1} (r-1-t) + (-1)^k k! \prod_{t=0}^{n-2-k} (r-1-t) \right) \quad (11)$$

and

$$0 = R(r) = \prod_{t=0}^{n-1} (r-2-t) - \sum_{k=[(n-1)/2]}^{n-2} \frac{a_{k, n-2-k}(z_0)}{f(z_0)} \left( (-1)^{n-2-k} (n-1-k)! \prod_{t=0}^{k-1} (r-2-t) + (-1)^k (k+1)! \prod_{t=0}^{n-3-k} (r-2-t) \right) \quad (12)$$

(here and further the product of empty set of terms, as well as  $0!$  are considered to be equal to 1).

Suppose that the initial equation (2) has the Painlevé property. Then, according to the theorem 1, for everywhere dense in the complex space set of  $z_0$  the corresponding resonance equation  $R(r) = 0$  has  $n - 1$  different positive integer roots in addition to the trivial root  $r = -1$ . Since the coefficients of the equation  $R(r) = 0$  as well as it's roots depend continuously on  $z_0$ , the above mentioned condition is possible only if these coefficients and roots are constant with respect to  $z_0$ .

Denote the positive integer roots of the corresponding resonance equation (11) or (12) by  $0 < r_1 < r_2 < \dots < r_{n-1}$  and let  $r_0 = -1$  be the trivial root.

## 4 Solving the resonance equation in case of Bureau number 2

According to the Viet theorem for (12) one can get

$$\sum_{j=0}^{n-1} (r_j - 2) = \sum_{j=0}^{n-1} j, \quad (13)$$

$$\sum_{j=0}^{n-1} (r_j - 2)^2 = \sum_{j=0}^{n-1} j^2 + 2h, \quad (14)$$

where  $h = a_{n-2,0}(z_0)/f(z_0)$  being constant with respect to  $z_0$ . Of course  $h$  should be integer. At the same time  $R(2) = (-1)^{n-1}(n-1)!h$ . If  $h > 0$  then  $R(2)$  and  $R(-\infty)$  have different signs, so an interval  $(-\infty; 2)$  can not contain an even number of roots  $r$  of the equation (12). Since the only non-positive root is  $r = -1$  this means that  $r = 1$  can not be another root. Also  $R(2) \neq 0$ , that is why  $2 < r_1 < r_2 < \dots < r_{n-1}$ . Let  $\delta_j = r_j - 2 - j$  for  $j = 1, 2, \dots, n-1$ . Then  $0 \leq \delta_1 \leq \delta_2 \leq \dots \leq \delta_{n-1}$  and, with respect to (13), one obtains  $0 = \sum_{j=0}^{n-1} (r_j - 2 - j) = -3 + \sum_{j=1}^{n-1} \delta_j$ . Then only the following three cases are possible:

- 1)  $\delta_{n-3} = \delta_{n-2} = \delta_{n-1} = 1, \delta_1 = \delta_2 = \dots = \delta_{n-4} = 0$ ;
- 2)  $\delta_{n-2} = 1, \delta_{n-1} = 2, \delta_1 = \delta_2 = \dots = \delta_{n-3} = 0$ ;
- 3)  $\delta_{n-1} = 3, \delta_1 = \delta_2 = \dots = \delta_{n-2} = 0$ .

However, with respect to (14) one can obtain

$$h = \left( 9 + \sum_{j=1}^{n-1} (\delta_j + j)^2 - \sum_{j=1}^{n-1} j^2 \right) / 2 = \left( 9 + \sum_{j=1}^{n-1} \delta_j^2 + 2 \sum_{j=1}^{n-1} j\delta_j \right) / 2. \quad (15)$$

From the other hand  $(-1)^{n-1}(n-1)!h = R(2) = (2 - (-1)) \prod_{j=1}^{n-1} (2 - (2 + j + \delta_j)) = 3(-1)^{n-1} \prod_{j=1}^{n-1} (j + \delta_j)$ , consequently

$$h = 3 \prod_{j=1}^{n-1} (j + \delta_j) / (n-1)! \quad (16)$$

In the case 1) with respect to (15) obtain  $h = (9 + 3 + 2(3n - 6)) / 2 = 3n$ . From the other hand, according to (15) obtain  $h = 3n/(n-3)$ . It is possible only if  $n = 4$ . Then the corresponding initial equation (4) takes form

$$w^{(IV)} = A(z) (ww'' + (w')^2) + a_1(z)w''' + a_2(z)w'' + a_3(z)w'w + a_4(z)w' + a_5(z)w^2 + a_6(z)w + a_7(z). \quad (17)$$

In the case 2) with respect to (15) obtain  $h = (9 + 5 + 2(2(n-1) + (n-2))) / 2 = 3n + 3$ . From the other hand, according to (15) obtain  $h = 3(n+1)/(n-2)$ . It is possible only if  $n = 3$ . Then the corresponding initial equation (4) takes form

$$w''' = A(z)ww' + a_1(z)w'' + a_2(z)w' + a_3(z)w^2 + a_4(z)w + a_5(z). \quad (18)$$

In the case 3) with respect to (15) obtain  $h = (9 + 9 + 2(3(n-1))) / 2 = 6 + 3n$ . From the other hand, according to (15) obtain  $h = 3(n+2)/(n-1)$ . It is possible only if  $n = 2$ . Then the corresponding initial equation (4) takes form

$$w'' = A(z)w^2 + a_1(z)w' + a_2(z)w + a_3(z). \quad (19)$$

If  $h < 0$  then assume  $\delta_j = r_j - 2 - j$  for  $j = 0, 1, \dots, n-1$  and according to (13), (14) obtain  $\sum_{j=0}^{n-1} \delta_j = 0$  and  $\sum_{j=0}^{n-1} (\delta_j)^2 + 2 \sum_{j=0}^{n-1} (j\delta_j) = 2h < 0$ . However since  $\delta_0 \leq \delta_1 \leq \dots \leq \delta_{n-1}$  one can obtain  $\sum_{j=0}^{n-1} (j\delta_j) > 0$

because  $\sum_{j=0}^{n-1} (j\delta_j) > \sum_{j=0}^{n-1} (j\delta_{n-1-j})$ , while  $\sum_{j=0}^{n-1} (j\delta_j) + \sum_{j=0}^{n-1} (j\delta_{n-1-j}) = (n-1) \sum_{j=0}^{n-1} \delta_j = 0$ . Consequently obtain the contradiction  $\sum_{j=0}^{n-1} (\delta_j)^2 = 2h - 2 \sum_{j=0}^{n-1} j\delta_j < 0$ .

Finally  $h = 0$  is not possible as  $a_{n-2,0}(z) \neq 0$ . This way in case of Bureau number 2 the only 3 possible forms for the equation (4) with the strong Painlevé property are (17)-(19).

## 5 Solving the resonance equation in case of Bureau number 1

First note that in case  $a_{n-1,0} = 0$  the equation (11) after normalization by  $r-1$  takes the form of (12) with order smaller by one and can be considered in the same way as above, which leads to the similar cases 1)-3) as for the equation (12). For these cases the initial equation (3) takes one of the three corresponding forms:

$$w^{(V)} = A(z) (w'w''' + (w'')^2) + a_1(z)w^{(IV)} + S(w''', w'', w', w, z), \quad (20)$$

$$w^{(IV)} = A(z)w'w'' + a_1(z)w''' + (a_2(z)w + a_3(z))w'' + a_4(z)(w')^2 + a_5(z)ww' + a_6(z)w' + a_7(z)w^2 + a_8(z)w + a_9(z), \quad (21)$$

$$w''' = A(z)(w')^2 + a_1(z)w'' + a_2(z)ww' + a_3(z)w' + a_4(z)w^2 + a_5(z)w + a_6(z), \quad (22)$$

where  $S$  is a quadratic form in  $w$  and its derivatives with coefficients locally analytic in  $z$  and cumulative number of derivatives in each term not exceeding 3.

Consider the case  $a_{n-1,0} \neq 0$  in assumption that the equation (11) possesses  $n-1$  distinct positive integer roots being a necessary condition for the strong Painlevé property. Let those roots be  $0 < r_1 < r_2 < \dots < r_{n-1}$  and also denote  $r_0 = -1$ . From the Viet theorem obtain

$$\sum_{j=0}^{n-1} r_j = \sum_{j=0}^{n-1} j + h, \quad (23)$$

where  $h = a_{n-1,0}(z_0)/f(z_0)$  being an integer non-zero constant. One can find  $R(1) = (-1)^n(n-1)!h \neq 0$ . Since the interval  $(-\infty; 1)$  contains a single non-multiple root  $r = -1$  of the equation (11), then  $R(1)$  and  $R(-\infty)$  should have different signs, consequently  $h < 0$ . Let  $\delta_j = r_j - 1 - j$  for  $j = 1, 2, \dots, n-1$ . Then  $0 \leq \delta_1 \leq \dots \leq \delta_{n-1}$  and according to (23) obtain  $-1 \geq h = r_0 - 1 + \sum_{j=1}^{n-1} (r_j - 1 - j) = -2 + \sum_{j=1}^{n-1} \delta_j$ .

So only following two cases are possible:

- 1)  $\delta_{n-1} = 1, \delta_1 = \delta_2 = \dots = \delta_{n-2} = 0$ ,
- 2)  $\delta_1 = \delta_2 = \dots = \delta_{n-1} = 0$ .

However from one side  $h = -2 + \sum_{j=1}^{n-1} \delta_j$  and from the other  $(-1)^n(n-1)!h = R(1) = 2(-1)^{n-1} \prod_{j=1}^{n-1} (j + \delta_j)$ . In the case 1) one can find  $h = -1$ ,  $(-1)^{n-1}(n-1)! = 2(-1)^{n-1} \prod_{j=1}^{n-1} (j + \delta_j) = 2(-1)^{n-1}(n-2)!n$  and consequently obtain a contradiction  $n-1 = 2n$ .

In the case 2) the conditions  $h = -2 + \sum_{j=1}^{n-1} \delta_j$  and  $(-1)^n(n-1)!h = R(1)$  correspond each other. The initial equation (4) takes the following form

$$w^{(n)} = A(z) (w^2)^{(n-1)} + \sum_{j+k < n-1} a_{k,j}(z)w^{(k)}w^{(j)} + \sum_{j=0}^{n-1} b_j(z)w^{(j)} + c(z). \quad (24)$$

## 6 The classification

So we have proven the following

**Theorem 2.** *If the equation (2) possesses the strong Painlevé property it should be of one of the following seven forms: (17)-(22), (24).*

Note that only one of these seven aforementioned equations — equation (24) — admits higher orders  $n \geq 6$ .

**Corollary 1.** *If the equation (2) of order  $n \geq 6$  possesses the strong Painlevé property it should be of the form (24).*

## 7 Necessary and sufficient conditions

The theorem 2 gives only the necessary conditions for the equation (2) to possess strong Painlevé property. To complete the Painlevé classification one should find the necessary and sufficient conditions of the strong Painlevé property for each of the equations (17)-(22), (24).

Note that the equations (17)-(19), (21), (22) are already studied and the conditions of Painlevé property for them are known. The equation (17) corresponds to the class F-I [4], the equation (18) — to the Chazy Class XIII [2], [3], the equation (19) — to the well-known case I(a) of the second order ([1], chapter XIV), equation (21) — to the class F-VII [5] and the equation (22) — to the Chazy Class I [2], [3].

Consider the equation (20). Note that by means of a variable substitution  $w = -12v/A(z)$ ,  $u = v''' + 6(v')^2 - K_1(z)v'' - K_2(z)v'v - K_3(z)v^2 - K_4(z)v' - K_5(z)v$  for the certain choice of  $K_1, K_2, \dots, K_5$  the equation (20) can always be transformed to a system of the form

$$\begin{cases} u'' = L_1(z)u' + L_2(z)u + L_3(z) + h_1(z)vv' + h_2(z)v' + h_3(z)v \\ v''' = 6(v')^2 + K_1(z)v'' + K_2(z)v'v + K_3(z)v^2 + K_4(z)v' + K_5(z)v + u, \end{cases} \quad (25)$$

where  $L_1, L_2, L_3, h_1, h_2, h_3$  are locally analytic functions in  $z$ .

**Lemma 1.** *If the equation (20) possesses the Painlevé property, then  $h_1(z) \equiv h_2(z) \equiv h_3(z) \equiv 0$ .*

Indeed, introduce the small parameter transform  $u = \alpha^{-4}U$ ,  $v = \alpha^{-1}V$ ,  $z = z_0 + \alpha x$ , where  $z_0$  is an arbitrary constant, and obtain the transformed system in the form

$$\begin{cases} U'' = \alpha L_1(z_0 + \alpha x)U' + \alpha^2 L_2(z_0 + \alpha x)U + \alpha^3 h_1(z_0 + \alpha x)VV' + \alpha^4 h_2(z_0 + \alpha x)V' + \\ \quad + \alpha^5 h_3(z_0 + \alpha x)V + O(\alpha^6), \\ V''' = -6(V')^2 + U + O(\alpha). \end{cases} \quad (26)$$

Consider the small parameter expansion for the solution of the system (26):

$$\begin{cases} U = U_0(x) + \alpha U_1(x) + \alpha^2 U_2(x) + \alpha^3 U_3(x) + \alpha^4 U_4(x) + O(\alpha^5), \\ V = V_0(x) + O(\alpha), \end{cases}$$

where  $U_0(x)$  is an arbitrary linear function, while  $V = V_0(x)$  is an arbitrary solution of the equation

$$V''' = -6(V')^2 + U_0(x), \quad (27)$$

Then  $U_1$  and  $U_2$  can be found as polynomials in  $x$ , while  $U_3''(x) = H_3(x) + h_1(z_0)V_0(x)V_0'(x)$ , where  $H_3$  is a polynomial in  $x$ .

So if the equation (20) possesses the Painlevé property then either  $h_1(z_0) = 0$ , or for any solution  $V = V_0(x)$  of the equation (27) the expression  $\int \int V_0(x)V_0'(x)dx dx$  should be single-valued so the function  $V_0(x)^2$  should always possess zero residue in any of its singularities. However, analyzing the possible Laurent series representation of general solution of (27) near movable pole one can find that this suggestion is invalid. So  $h_1(z_0) = 0$  for arbitrary  $z_0$ , consequently  $h_1(z) \equiv 0$ .

Further, in case  $h_1(z) \equiv 0$ , one can find  $U_4''(x) = H_4(x) + h_2(z_0)V_0'(x)$  where  $H_4$  is a polynomial in  $x$ . So if  $h_2(z_0) \neq 0$ , the expression  $\int V_0(x)dx$  should be single-valued for any solution  $V = V_0(x)$



of the equation (27). However this expression is multi-valued near the first order movable poles of  $V_0$ . Consequently  $h_2(z) \equiv 0$ . Finally  $h_3(z) \equiv 0$  in the same way, since  $U_5''(x) = H_5(x) + h_3(z_0)V_0(x)$  while the expression  $\int \int V_0(x) dx dx$  is also multi-valued near the first order movable poles of  $V_0$ . The proof of lemma 1 is now complete.

Now from lemma 1 one can see that the equation (20) with the Painlevé property should necessary possesses the second integral

$$v''' = -6(v')^2 + K_1(z)v'' + K_2(z)v'v + K_3(z)v^2 + K_4(z)v' + K_5(z)v + u(z), \quad (28)$$

where  $u(z)$  contains two constants of integration being an arbitrary solution of the second order linear equation

$$u'' = L_1(z)u' + L_2(z)u + L_3(z). \quad (29)$$

The integral (28) is the Chazy Class I equation [2], [3] and the necessary and sufficient conditions of the strong Painlevé for it are well-known:  $K_1(z) \equiv K_2(z) \equiv 0$ ,  $K_3(z) \equiv K_4(z)$ ,  $K_4''(z) \equiv (K_4(z))^2$ ,  $K_5''(z) \equiv K_4(z)K_5(z)$  and  $u''(z) \equiv K_4(z)u(z)/3 + (K_5(z)/6)^2$ , i.e.  $L_1(z) \equiv 0$ ,  $L_2(z) \equiv K_4(z)/3$ ,  $L_3(z) \equiv (K_5(z)/6)^2$ .

Finally consider the equation (24).

**Theorem 3.** *The equation (24) possess the strong Painlevé property if and only if it is linearizable by means of the variable change*

$$u = w' - A(z)w^2 - B(z)w, \quad (30)$$

where  $B(z)$  is a certain locally analytic function.

Of course if the equation (24) is linearizable by means of (30) then the equation (24) possess the strong Painlevé property, since this way the general solution  $u$  of the linear differential equation is free of any movable singularities, while the correspondent function  $w$  can be found by resolving the Riccati equation (30) and so all of it's movable singularities are poles.

To prove the inverse statement consider the equation (24) and assume that it possesses the strong Painlevé property and consequently the Painlevé property. Introduce the variable change (30) where  $B(z)$  is a locally analytic function, undefined yet. Then the equation (24) can be transformed to:

$$\begin{aligned} u^{(n-1)} &= \sum_{p(\chi) \leq n} \tilde{a}_\chi(z) w^{\chi_0} \prod_{j=1}^{n-1} (u^{(j-1)})^{\chi_j}, \\ w' &= u + A(z)w^2 + B(z)w, \end{aligned} \quad (31)$$

where  $\chi = (\chi_0, \chi_1, \dots, \chi_{n-1})$  are multi-indexes with integer non-negative components,  $p(\chi) = \sum_{j=0}^{n-1} (j+1)\chi_j$ , and  $\tilde{a}_\chi$  are locally analytic in  $z$  coefficients which can be polynomially expressed in terms of the coefficients of the initial equation (24), functions  $A, B$  and their derivatives. Consider the coefficient  $\tilde{a}_{(0,0,\dots,0,n)}$  at  $w^n$  in the right-hand side of the first equation of the system (31). It depends linearly on  $B(z)$  being of the form  $-n!B(z)A(z)^n + \dots$ , where dots denote terms not containing  $B(z)$ . Consequently by the corresponding choice of  $B(z)$ , one can always make the degree of the system (31) first equation's right-hand side to be not higher than  $n-1$  with respect to  $w$ .

Demonstrate that in this case the right-hand side of the first equation of the system (31) should not depend on  $w$  at all. Indeed, suppose the opposite case. Then by means of a small parameter transform  $z = z_0 + \alpha x$ ,  $u = \alpha^{-2}U$ ,  $w = \alpha^{-1}W$  with arbitrary constant  $z_0$ , the system (31) is transformed to

$$\begin{cases} \frac{d^{n-1}U}{dx^{n-1}} = \alpha T(U^{(n-2)}, U^{(n-3)}, \dots, U, x, \alpha) + \alpha^k H(W, U^{(n-2)}, U^{(n-3)}, \dots, U) + o(\alpha^k), \\ \frac{dW}{dx} = U + A(z_0)W^2 + O(\alpha), \end{cases} \quad (32)$$

where  $k$  is a certain positive integer, while  $T, H$  are polynomials in all variables, while  $1 \leq \deg_W H \leq n-1$ .

The solution of the system (32) can be found in the form

$$\begin{aligned}
U(x, \alpha) &= U_0(x) + \sum_{j=1}^{k-1} U_j(x) \alpha^j + \alpha^k \int \int \dots \int H(W_0(x), U_0^{(n-2)}(x), U_0^{(n-3)}(x), \dots, U_0(x)) dx^{n-1} + \\
&\quad + o(\alpha^k), \\
W(x, \alpha) &= W_0(x) + O(\alpha),
\end{aligned} \tag{33}$$

where  $U_0(x) = C_{n-2}x^{n-2} + C_{n-3}x^{n-3} + \dots + C_1x + C_0$  and  $W_0$  is an arbitrary solution of the Riccati equation  $W' = A(z_0)W^2 + U_0(x)$  with a movable simple pole in  $x = -C$ , while  $C, C_0, C_1, \dots, C_{n-2}$  — are arbitrary complex constants. One can see that the function  $U(x, \alpha)$ , being determined by (33) is multi-valued in some neighborhood of the point  $x = -C$  for sufficiently close to zero nonzero  $\alpha$ , because the function  $H(W_0(x), U_0^{(n-2)}(x), U_0^{(n-3)}(x), \dots, U_0(x))$  in general case admits a pole of order  $\deg_W H \leq n-1$  in  $x = -C$ .

Consequently (30) transforms the equation (24) to the polynomial differential equation of order  $n-1$  in  $u$ , not depending on  $w$ . In case this equation is nonlinear, then according to the theorem 4 [15] its Bureau number should be 1 or 2 so for at least one of the terms the inequality  $p(\chi) \geq n+1$  should hold. However for all terms we have  $p(\chi) \leq n$ . This contradiction completes the proof of the theorem 3.

This way the necessary and sufficient conditions of the strong Painlevé property for each of the possible seven cases (17)-(22), (24) are constructed completing the strong-Painlevé classification for the second degree arbitrary order polynomial equations (2). Note that the only possible equation (2) of order  $n \geq 6$  with the strong Painlevé property, i.e. the equation (24), is linearizable, while others could be transformed to the equations previously known. In particular this means that solutions of the second degree polynomial differential equations having the strong Painlevé property do not provide any new transcendental functions.

## Conclusions

We've built a complete classification of the second degree arbitrary order polynomial ordinary differential equations (2) having strong Painlevé property. We proved that all such equations are contained in 7 classes (17)-(22), (24), and for each of those classes necessary and sufficient conditions for the strong Painlevé property are obtained. Six classes (17)-(22) happen to have a limited order  $n \leq 6$  and if having a strong Painlevé property all appear to be reduced to the previously known equations, while the only class (24) of unlimited order appears to be linearizable. This way it is proven that second degree polynomial equations (2) of the arbitrary order having the strong Painlevé property are all integrable by means of known functions and do not provide any new transcendental solutions.

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